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AND
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"STRUCTURE OF THE WEAK STABILITY BOUNDARY III"

Science Mission Directorate

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SUMMARY

This report represents a summary of the main results obtained within the third year(2008-09) of a three year research project(0607,0708,0809) in

NASA's AISR program, Science Mission Directorate, with grant number NNG06GG55G. The main goal for this research is to find ways for spacecraft to remain in orbit about a planet for extended periods of time, moving in a more flexible manner and performing maneuvers using substantially less fuel than by standard methods. This would provide more flexible maneuvering and a way to extend mission duration and therefore data collection. The methodology utilizes very sensitive motion for spacecraft that occurs about planetary bodies when the velocity is suitably adjusted to special values. The region supporting this sensitive motion is called a weak stability boundary. Although the resulting motion is unstable in such circumstances, the instability itself is used to reduce the fuel (or equivalently the change in velocity, DV) to change the orbital motion.

The weak stability boundary was first estimated in 1986 by this researcher [1, 2], and it is a complicated region defined in position-velocity space. It was first defined about the Moon, which we consider here. The structure of this region has not been understood since then, and there have been some results shedding light on its nature. Earlier work done by this researcher in 1990 indicated that the motion associated with this region was both unstable, chaotic in nature. It also had the property that trajectories which emanated from this boundary, and which escaped the Moon, would move in an elliptic-type orbit about the Earth that was in resonance with the Moon. Also, subsequent encounters with the boundary by a trajectory would cause the trajectory to abruptly jump to different resonance motions. These properties of the weak stability boundary made it clear that its mathematical structure was very complicated. Although it is not understood at this time, the work performed thus far in this project has helped shed a lot of light on the nature of this region. The first two years of this project were more focused on the resonance dynamics. These results are summarized in previous reports, [3, 4]. The details of these results are published in [5].

This report summarizes new results which open the door to understanding this region for the first time and are very intriguing. A key insight is some interesting work by F. Garcia and G. Gomez [10] in 2007. They demonstrated that the definition of the weak stability boundary about the Moon could be generalized. The initial definition of this region by Belbruno made use of monitoring the stability of a trajectory about the Moon after 1 cycle. Garcia and Gomez suggested that this be generalized to n-cycles, $n = 1, 2, 3, ...$ This gives rise to the nth weak stability boundary. The general boundary is then obtained by taking a union of these n boundaries. This generalized boundary can then be visualized and it has an interesting fractal appearance. This work was studied further in [12] and is the subject of this report. The generalized boundary is visualized for different parameter values. It is conjectured, but not demonstrated in [12], that the generalized weak stability boundary may be related to a special set, which is the limit set resulting from the invariant manifolds of the Lyapunov orbits associated to L_1, L_2 near the Moon. More precisely, this limit set may actually be equivalent to the weak stability for certain parameter values. If true, that would be a surprising result and provide a new approach to studying such limit sets in general. It would also provide the first precise description of the structure this region. In this way, this boundary provides a straight forward way of using local information to estimate a complicated global limit set that would normally require extensive numerical computation, promising a number of applications to mission design and dynamical astronomy.

1. INTRODUCTION

The search for a new type of transfer from the Earth to the Moon for spacecraft in 1986 led to the discovery of an interesting region of unstable motion about the Moon [2]. The motivation was to find a way for a spacecraft to arrive near the Moon with a substantially reduced relative velocity as compared to the standard Hohmann transfer. A Hohmann transfer approaches the Moon with a relative velocity of approximately 1 km/s, resulting in a significant amount of fuel being required to slow down and go into lunar orbit. It was desired to reduce the approach velocity to 0 km/s. This is called ballistic capture. Although such a capture was conjectured in the 1960's by C. Conley [2, 9] in the three-body problem between the Earth-Moon-spacecraft for transfers starting from arbitrarily near to the Earth to near the Moon, it had never been demonstrated. It was suggested from Conley's work that the invariant manifold structure associated to the unstable collinear Lagrange points L_1, L_2 near the Moon would have to somehow play a role, but this was not understood.

A way to achieve ballistic capture was numerically demonstrated in 1986 for transfers starting sufficiently far from the Earth [1, 2]. The solution to this problem, yielding a transfer to the Moon for a spacecraft using low thrust, with a flight time of two years, utilizes a region about the Moon where the stability of motion is in transition. This is where a particle, say a spacecraft, is in between capture and escape with respect to the Moon. The capture, defined by a negative Kepler energy, is temporary, termed weak capture. The region about the Moon where weak capture occurs is defined by the weak stability boundary (WSB). It can be estimated by a numerical algorithm which determines the transition between 'stable' and 'unstable' motion about the Moon. Stable and unstable motion in this case are associated to whether or not a spacecraft can, or cannot, respectively, perform a complete cycle about the Moon.

A transfer to the Moon arriving in ballistic capture can be achieved by arriving in weak capture, or equivalently at the WSB. It turned out that the solution obtained in 1986 did utilize the dynamics near the invariant manifolds associated with the Lyapunov orbits associated to the collinear Lagrange points. The full solution to Conley's conjecture was to find a transfer arriving at the lunar WSB starting from an *arbitrary* distance from the Earth instead of sufficiently far away. This was accomplished in 1991 with the rescue of a Japanese lunar mission and getting it's spacecraft, *Hiten*, to the Moon with very little fuel on a new type of transfer. This solution utilized a four-body problem between the Earth-Moon-Sun-spacecraft [1, 2]. The manifold structure associated to the dynamics of that transfer was partially uncovered in 1994 [1]. This was further explored in 2000 by G. Marsden, et al [11]. Also, see the work by C. Circi and P. Teofilatto [8]. The type of transfer that *Hiten* used promises to play an important role in future lunar missions.

Up to recently, the nature of the weak stability boundary, and associated dynamics, has not been well understood. One of the main results here is to breifly describe recent work which sheds light on this problem. We will consider the restricted three-body problem for the motion of a particle P_3 of zero mass in a gravitational field generated by two primary particles P_1, P_2 in mutual circular motion, where the mass of P_1 is much larger than the mass of P_2. We will assume for this report, that P_1, P_2 are the Earth, Moon, respectively, while P_3 is a spacecraft. Also, it is assumed that P_3 is constrained to move on the same plane as P_1, P_2. The motion of P_3 is studied relative to P_2, and the weak stability boundary exists about P_2.

We'll first describe the orginal alogrithmic definiton of the weak stability boundary in 1986 and then describe a new generalized definition and the resulting visualizations obtained of this region. This provides a way to accurately visualize this boundary for the first time. We will also mention how this work suggests a connection between the weak stability boundary and the limit set obtained from the invariant manifolds associated to the Lyapunov orbits about L_1, L_2 near P_2. This connection is not yet understood and is the focus of future work. The main references for these results are [6, 10, 12].

The WSB region is is not an invariant set, nor does it lie on individual energy levels which has made its study difficult. However, these results promise to help better understand this region.

Restricted Three-Body Problem Model

The model we will use until further notice is the restricted three-body problem mentioned in the Introduction between particles P_1, P_2, P_3. This is a model that provides a way to study dynamics in the three-body problem where the problem has been simplified as much as possible and still preserves the three-body interaction. Although this more simplified version of the three-body problem is being used, it is found that the results obtained are close to a more realistic three-body modeling. We view this problem in a coordinate system x, y which rotates with the same velocity as P_1, P_2 about their common center of mass cm. This is called a rotating coordinate system, and in it, P_1, P_2 are fixed. For convenience, we put P_1, P_2 on the x-axis, and the cm at the origin, where P_1 is at $(\mu, 0)$ and P_2 at $(-1+\mu, 0)$, $\mu = m_2/(m_1+m_2) \approx m_2/m_1 \approx .012$, is the mass ratio of the Moon to the Earth, m_1, m_2 are the mass of the Earth, Moon, respectively. In these units, the mass of the Earth is $1-\mu$ and the mass of the Moon is μ. This coordinate system is shown in Figure 1. The Lagrange points $L_k, k = 1, 2, 3, 4, 5$ are also shown. The differential equations for the motion of P_3 are given by,

$$\begin{aligned} \ddot{x} - 2\dot{y} &= x + \Omega_x \\ \ddot{y} + 2\dot{x} &= y + \Omega_y, \end{aligned} \tag{1}$$

where $\dot{} \equiv \frac{d}{dt}, \Omega_x \equiv \frac{\partial \Omega}{\partial x}$,

$$\Omega = \frac{1-\mu}{r_1} + \frac{\mu}{r_2},$$

r_1 = distance of P_3 to $P_1 = [(x-\mu)^2 + y^2]^{\frac{1}{2}}$, and r_2 = distance of P_3 to $P_2 = [(x+1-\mu)^2 + y^2]^{\frac{1}{2}}$, see Figure 1. We note that the the coordinates are scaled with no dimension. To obtain dimensional units, in kilometers(km) for distance, and seconds (s) for time, the dimensionless position (x, y) is multiplied by the distance, d, in kilometers between $P_1 - P_2$ and the velocity $\dot{x}, \dot{y}$ is multiplied by the circular velocity, vc, in kilometers per second of P_2 about P_1. For example if $P_1, P_2 =$ Earth, Moon, respectively, then $d = 386,000$km, $vc = 1$km/s.

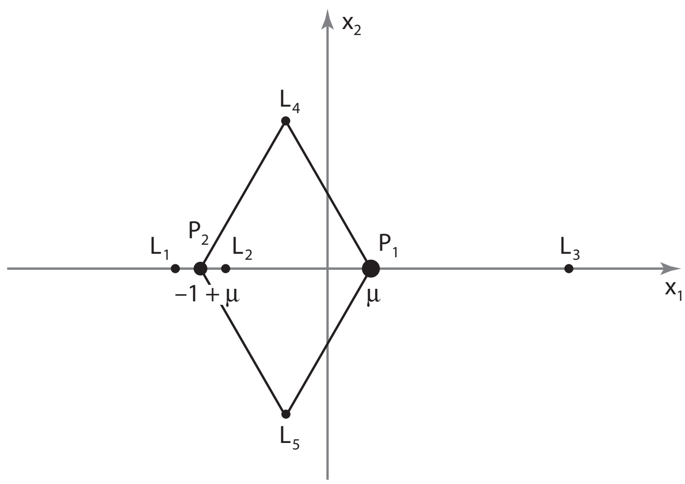

Figure 1: Rotating coordinate system and locations of the Lagrange points.

System (1) of differential equations has five equilibrium points, where $\ddot{x} = \ddot{y} = 0$, and $\dot{x} = \dot{y} = 0$, which are the Lagrange points $L_k, k = 1, 2, 3, 4, 5$. Placing P_3 at any of these locations implies it will remain fixed at these positions for all time. Three of these points are called collinear and lie on the x-axis, and the two that lie off of the x-axis are called equilateral points.

The total energy of the system is called the Jacobi energy, J, which is a function of $(x, y, \dot{x}, \dot{y})$. It is given by

$$J = -(\dot{x}^2 + \dot{y}^2) + (x^2 + y^2) + \mu(1 - \mu) + 2\Omega. \tag{2}$$

J is an integral of the motion of (1). Thus, for any solution $\psi(t) = (x(t), y(t), \dot{x}(t), \dot{y}(t))$ of (1), $J(\psi(t)) = C =$ constant. C is called the Jacobi constant.

2. DETERMINATION OF THE WEAK STABILITY BOUNDARY

The weak stability boundary region was first estimated in 1986 as a way for spacecraft to be ballistically captured into orbit about the Moon, defined where the Kepler energy, H_2, with respect to the Moon, P_2, is negative,

$$H_2 = \frac{1}{2}v_2^2 - \frac{\mu}{r_2} < 0, \tag{3}$$

v_2 is the magnitude of the velocity of P_3 with respect to P_2. This gives rise to weak capture where P_3 will remain captured about P_2 for a finite time, t, $t_1 \leq t \leq t_2$, and where $H_2 > 0$ for $t < t_1, t > t_2$, i.e. where it escapes P_2.

The estimation of this region was originally accomplished by a numerical algorithm which measured when P_3 was able to perform a complete cycle about P_2 with initial elliptic conditions on a radial line l centered at P_2 and returning to l. This was first done in 1986 [2] then more rigorously in [1]. More precisely, the initial conditions on l assume, therefore, that $H_2 < 0$, or equivalently, where the initial eccentricity e_2 of P_3 with respect to P_2 satisfies $e_2 < 1$ at the initial time $t = 0$. A value of $e_2 \in [0, 1)$ is fixed. It is assumed that the initial velocity vector on l is normal to the line, in the posigrade direction, and that the initial state is at the periapsis of an osculating ellipse. Thus,

$$v_2 = \sqrt{\frac{\mu(1 + e_2)}{r_2}}. \tag{4}$$

We assume l makes an angle $\theta_2 \in [0, 2\pi]$ with the $x-$axis, indicated by $l(\theta_2)$, which is fixed. With a given initial state for P_3 at $t = 0$, the differential equations given by (1) are numerically integrated for $t > 0$.

If the trajectory for P_3 in position space, $\alpha(t) = (x(t), y(t))$, performs a full cycle about P_2 and returns to l with $H_2 < 0$, then the motion is called *stable*. If, on the other hand, P_3 returns to l with $H_2 \geq 0$, or if $\alpha(t)$ moves away from P_2 and makes a full cycle about P_1, then the motion is called *unstable*. (See Figure 2.)

By iterating between stable and unstable motion, one finds a critical distance r^* on l with the property that for $r_2 < r^*$ the motion is stable and for $r_2 > r^*$ the motion is unstable. Since r^* depends on θ_2 and e_2, that are held fixed during the iteration process, we obtain a functional relationship

$$r^* = f(e_2, \theta_2) \tag{5}$$

which defines the weak stability boundary about P_2, we label $\mathcal{W}$.

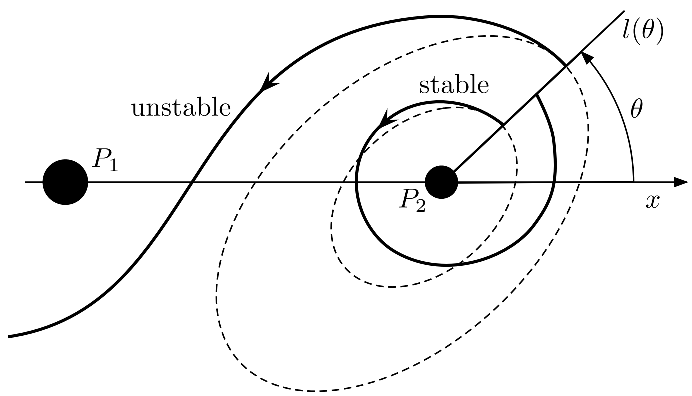

Figure 2: Stable and unstable motion after one cycle about P_2.

The definition of the weak stability boundary given by $\mathcal{W}$ can be generalized to find a more accurate definition of this transition region. This is because the transition distance given by (5) is not unique. This generalization was given by Garcia and Gomez [10] and shows that the weak stability boundary is much more complicated than originally thought. Their analysis was studied by Topputo and Belbruno and additional refinements were obtained [12]. Some of the results of these studies are summarized here and the reader can find many more details in these papers.

It was found that for a given value of θ_2 and e_2, there are a countable number of open intervals, $I_k = (r^*_{2k-1}, r^*_{2k})$, $k = 1, 2, 3, ...$, $r^*_1 = 0$, containing stable points along l and that the points defining the transition between stable and unstable motion, which define the weak stability boundary, lie at the boundaries of these open intervals. The stable set of points, $U_1(e_2, \theta_2)$, is therefore given by,

$$U_1(e_2, \theta_2) = \bigcup_{k \geq 1} I_k. \tag{6}$$

The more general definition of the weak stability boundary as e_2, θ_2 vary is given by the set of boundary points of this set, except r^*_1. We label this

$$\mathcal{W}_1 = \partial \bigcup_{e_2 \in [0,1), \theta_2 \in [0, 2\pi]} U_1(e_2, \theta_2), \tag{7}$$

where, for a set A, ∂A represents the boundary of A.

Equation 7 yields a definition of the weak stability boundary when studying trajectories that make one cycle about P_2. We refer to this as the weak stability boundary of order one. Analogously, U_1 is referred to as a set where the points are 1-stable. More generally, a similar definition can be made after analyzing n cycles about P_2 before returning to l, $n = 1, 2, 3, ...$ Thus, the weak stability boundary of order $n = 1, 2, 3, ...$, relative to n cycles of P_3 about P_2, is given by

$$\mathcal{W}_n = \partial \bigcup_{e_2 \in [0,1), \theta_2 \in [0, 2\pi]} U_n(e_2, \theta_2), \tag{8}$$

where U_n consists of points that we refer to as n-stable,

$$U_n = \bigcup_{k \geq 1} I_k \tag{9}$$

Summarizing, the weak stability boundary of order n, denoted by $\mathcal{W}_n$ is the locus of all points $r^*(e_2, \theta_2)$ along the radial segment $l(\theta_2)$ for which there is a change of stability of the initial trajectory $\alpha(t)$, that is, $r^*(e_2, \theta_2)$ is one of the endpoints of an interval $I_k = (r^*_{2k-1}, r^*_{2k})$ characterized by the fact that for all $r \in I_k$ the motion is n-stable and there exist $r' \notin I_k$, arbitrarily close to either r^*_{2k-1} or r^*_{2k} for which the the motion is n-unstable. Thus

$$\mathcal{W}_n = \{ r^*(e_2, \theta_2) \,|\, e_2 \in [0, 1), \theta_2 \in [0, 2\pi] \}.$$

We can define a subset of the weak stability boundary of order n, $\mathcal{W}_n(e_2)$, obtained by fixing the eccentricity of the osculating ellipse,

$$\mathcal{W}_n(e_2) = \{ r^*(e_2, \theta_2) \,|\, \theta_2 \in [0, 2\pi] \}. \tag{10}$$

The sets $\mathcal{W}_n$, U_n are graphically determined in [10, 12] for many different parameter values. This takes a substantial amount of numerical work and the details can be found in these papers. For the sake of brevity just two figures are displayed from [12]. Figure 3 shows the weak stability boundary of order 1 for the case $e_2 = 0$ as the boundary of the set U_1 of 1-stable points.

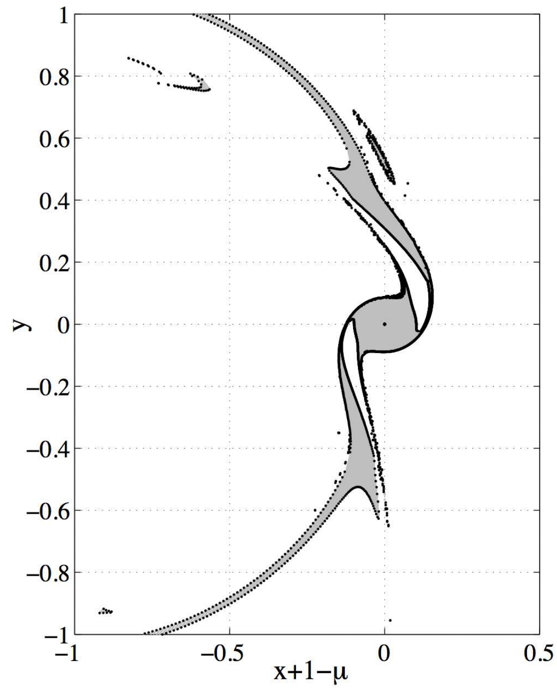

Figure 3: The open set U_1 of 1-stable points for the case $e_2 = 0$ with boundary $\mathcal{W}_1(0)$ centered at P_2. Multiple components of the boundary are shown.

Multiple components are shown indicating a complicated structure. Figure 4 shows the sets U_n with respective boundaries $\mathcal{W}_n$, $n = 1, ..., 6$ for $e_2 = 0$. These sets become more sparse as n increases.

Results in [12] show that the size of the sets U_n, and hence $\mathcal{W}_n$, reduce in size as $e_2 \to 1$. It is also seen that these sets exist for a certain range of the Jacobi constant.

Preliminary results obtained in [12] indicate that the weak stability boundary of order n is related to the invariant manifold structure associated with the limit sets of the stable manifolds associated to the Lyapunov orbits near L_1, L_2 for a specific range of Jacobi constant. This is currently being studied by F. Topputo, M. Gidea and this author. Figure 5 shows trajectories with initial conditions on $\mathcal{W}_1(0)$ which in forward time move near to the stable manifold on the Lyapunov orbit about L_1.

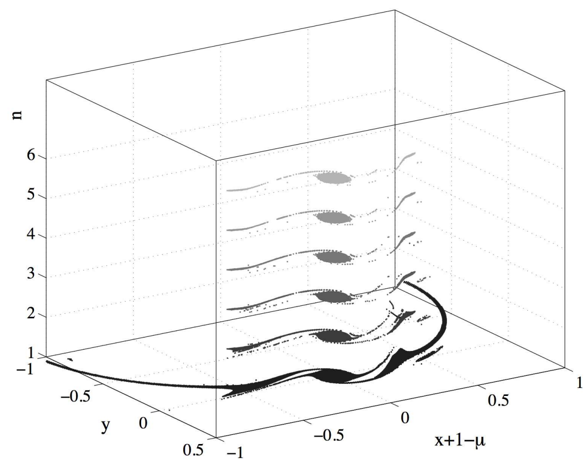

Figure 4: n-stable sets and nth order weak stability boundaries for $e_2 = 0$, $n = 1, ..., 6$.

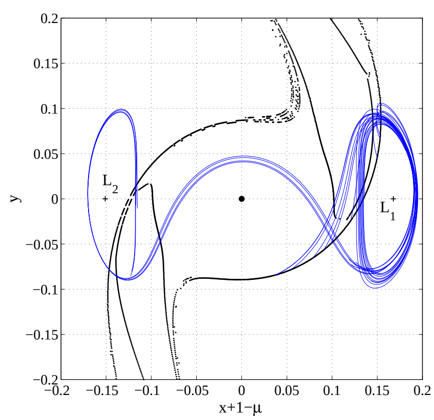

Figure 5: Trajectories with initial conditions on $\mathcal{W}_1(0)$ for $\theta_2 \in [-\pi/4, \pi/4]$ moving close to the stable manifold of the Lyapunov orbit about L_1

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